ON THE THEORY OF OPTIMUM AVERAGING OF DYNAMIC SYSTEMS CONTROLS

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The completeness of controlled systems optimization depends on the level of the information available about their operation. Measurements and predictions about the state of the system and the medium with which it interacts, as well as the control signals, inevitably contain uncontrolled components as, for example, random instrument errors, disturbances, etc. Thus, information on the past, present, and future of systems can be complete only to the extent that data about random events are complete. Hence, the optimization of controlled systems is not in the final analysis reducible to the optimum averaging of their controls in the sense that one must necessarily be concerned with the appropriate formation of the average control signal value, which must be the same for all states attainable by the system as long as the latter do not exceed certain limits, e.g. the dead zones of the measuring devices. The optimization criterion must provide for the attainment of the extremum by one of the averaged system characteristics [1 and 2].

We should also bear in mind another aspect of the question, namely the fact that certain systems are designed for use under various conditions and for the fulfillment of different tasks while employing the same control algorithm. Here we must naturally see to it that the control minimizes the average loss engendered under the various conditions of system operation [3].

The problem of optimum averaging of controls has several different aspects. Essentially, it can be made to encompass the deterministic formulation of the problem as one which corresponds to the minimum level of utilization of information about the true conditions of process realization as represented solely in terms of the mathematical expectations of the determining functions and parameters.

We shall develop the notion of optimum controls averaging as conceived in [1 to 4] and suggest a general method of solving problems on the basis of the principle of optimum controls averaging (Theorem 2) which we shall prove. This opens the way for the formulation and solution of new practical problems.

Thus, in Section 4 we shall analyze the previously untreated problem of optimizing programed control systems "as a whole", i.e. with the undisturbed motion and the disturbed motion control law optimized in accordance with a single criterion with allowance for their interrelationships.

The mechanical basis and one of the possible areas of application of the theory to be developed are also characterized by the following stochastic variant proposed in [5] wherein the object is to find the reactive acceleration $u = u(t)$ of a point of variable mass moving in a forceless field with a constant expenditure of energy and a minimum value of the functional

$$J_0 = \int_{t_0}^{t_1} u^2 dt \tag{0.1}$$

under the conditions that the position $x_1 = x_1(t)$, velocity $x_2 = x_2(t)$, and the point control objective are described by Equations

$$x_1^{\cdot} = x_2, \qquad x_2^{\cdot} = u, \qquad x_1(t_0) = x_{10}, \qquad x_2(t_0) = x_{20} \tag{0.2}$$

$$x_1(t_1) = x_{11}, \qquad x_2(t_1) = x_{21} \tag{0.3}$$

Here t_0, t_1, x_{i0}, x_{i1} $(i = 1,2)$ are specified. In reality x_1, x_2 are usually random functions of time, $x_1 = X_1(t)$, $x_2 = X_2(t)$, defined by equations which replace system (0.2), (0.4)

$$X_1^{\cdot} = X_2, \qquad X_2^{\cdot} = u + u^l + \xi_1 + \xi_2, \qquad X_1(t_0) = x_{10} + L_1, \qquad X_2(t_0) = x_{20} + L_2$$

Here L_1, L_2 are random perturbations in the parameters of the initial state of the point; u^l is the reactive acceleration produced by the corrective arrangement in accordance with the realizations of L_1, L_2 and intended to compensate for the consequences of random perturbations in the parameters of the point's initial state; $\xi_i = \xi_i(t)$ is the error involved in the reproduction of the reactive acceleration. It is a random function of time which can be represented in the form of a canonical expansion [6] with determined coordinate functions $\xi_{ij}(t)$ and random coefficients P_{ij}.

Since the control signals u and u^1 are formed independently of the realizations of $\xi_i(t)$, it follows that exact fulfillment of boundary conditions of the form (0.3), i.e. $X_1(t_1) = x_{11}$, $X_2(t_1) = x_{21}$ is impossible. Thus, from conditions (0.3) we must seek fulfillment of the relations

$$M(X_{11} \mid L_1, L_2) = x_{11}, \qquad M(X_{21} \mid L_1, L_2) = x_{21}$$

or

$$M(X_{i1} - x_{i1} \mid L_1, L_2) = 0 \qquad (i = 1, 2) \qquad (X_{i1} = X_i(t_1)) \tag{0.5}$$

Here $M(X_{i1} - x_{i1} \mid L_1, L_2)$ is the conditional mathematical expectation of the random quantity $X_{i1} - x_{i1}$.

The control quality can be naturally evaluated by way of the functional

$$J = M\left\{\int_{t_0}^{t_1} [(u + \xi_1)^2 + \alpha(u^l + \xi_2)^2]\, dt\right\} \tag{0.6}$$

Introducing the function $X_3(t)$ as determined from Equations

$$X_3^{\cdot} = (u + \xi_1)^2 + \alpha(u^l + \xi_2)^2, \qquad X_3(t_0) = 0 \tag{0.7}$$

we obtain

$$J = M(X_{31}) \qquad (X_{31} = X_3(t_1), \ \alpha = \text{const} > 0) \tag{0.8}$$

We now see that the example of [5] in the stochastic variant which we have been considering is reducible to the solution of the following specific optimum problem: for system (0.4), (0.7) we are to find control functions $u = u(t)$, $u^l = u^l(t, L_1, L_2)$, which minimize functional (0.8) with boundary conditions (0.5). It is easy to see that this problem is a particular case of the problem whose formulation and solution,is the subject of the present paper.

1. Let there be a controlled system designed to realize the functionals

$$J_i = f_i(\mathbf{X}_1, a, a^l, T^l, P, L) \qquad (i = 0, \ldots, k_0) \tag{1.1}$$

defined by the equations of motion

$$X_i^{\cdot} = \varphi_i(t, X, v, v^l, P, L), \quad X_i(t_0^l) = \psi_i(P, L), \quad \mathbf{X}_1 = [X_{i1} = X_i(t_1^l)]$$
$$(t_0^l \leqslant t \leqslant t_1^l;\ i = 1, \ldots, n) \tag{1.2}$$

Here $X = (X_1, \ldots, X_n)$ is a continuous random vector-function;

$$T^l = (t_0{}^l, t_1{}^l), \qquad v = [a = (a_i), u = (u_j)] \quad (i = 1, \ldots, m; \ j = 1, \ldots, r)$$

$$v^l = [a^l = (a_i{}^l), u^l = (u_j{}^l)] \quad (i = 1, \ldots, m_0; \ j = 1, \ldots, r_0)$$

are the control parameters; a = const , and $u = u(t)$, $u^l = u^l(t, L)$, $a^l = a^l(L)$, $T^l = T^l(L)$ are from the open kernel of the domain U of bounded piecewise-continuous piecewise-smooth functions with a finite number of discontinuities; f_i, φ_i, ψ_i, f and their first and second derivatives are continuous functions; $P = (P_i)$, $L = (L_i)$ are random parameters determined in the domains Ω_p and Ω_l, respectively, by the distribution density $f = f_p(p \mid l) f_l(l)$, where $f_p(p/l)$ is an arbitrary law of P distribution.

Let us emphasize the difference between the controls v and v^l, T^l: the former is formed independently of the realizations of P, L, defining the program of motion and the nominal structural parameters of the system; the second does not depend on P but takes into account the realizations of L and thereby possesses controlling properties.

As we see, functionals (1.1) are random quantities whose characteristics are determined not only by the values of P and L, but also by the form of the control v, v^l, T^l. The purpose of many controlled systems can be expressed in the form of Equations

$$M[f_i(\mathbf{X}_1, a, a^l, T^l, P, L)] = 0, \quad M[f_j(\mathbf{X}_1, a, a^l, T^l, P, L) \mid l] = 0 \quad (1.3)$$
$$(i = 1, \ldots, k; \ j = k+1, \ldots, k_0)$$

and their quality evaluated by means of the functionals

$$J^l = M[f_0(\mathbf{X}_1, a, a^l, T^l, P, L) \mid l] \qquad (1.4)$$

$$J = M[f_0(\mathbf{X}_1, a, a^l, T^l, P, L)]] \qquad (1.5)$$

Here M is the symbol for the mathematical expectation; $M(f_i/l)$ is the conditional mathematical expectation of the random quantity $f_i(X_1, a, a^l, T^l, P, L)$.

If there exists a family of permissible controls of system (1.2) (i.e. controls which satisfy (1.3)), then we can pose the problem of choosing that control which minimizes functional (1.5). This control can be called average-optimum control, and the process leading to its determination referred to as the solution of the problem of optimum averaging of the control for system (1.2).

2. The possibility of constructing a family of permissible controls and the principle of optimum averaging of the control v, v^l, T^l are established by two theorems.

Theorem 2.1. If v, v^l, T^l is a permissible control, then there exists a family of permissible controls which includes v, v^l, T^l, if the ranks of the matrices

$$A = \begin{Vmatrix} B_{11} & \ldots & B_{1,\varkappa+m} \\ \cdot & \cdot & \cdot \\ B_{k1} & \ldots & B_{k,\varkappa+m} \end{Vmatrix}, \quad A^l = \begin{Vmatrix} B_{11}{}^l & \ldots & B_{1y}{}^l \\ \cdot & \cdot & \cdot \\ B_{y1}{}^l & \ldots & B_{yy}{}^l \end{Vmatrix} \qquad (2.1)$$

are equal to k and $y = k_0 - k$, respectively.

Here

$$B_{ij} = M\left[\sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial \alpha_j} + \sum_{\gamma=1}^{y} \frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\ l}} \frac{\partial \alpha_\gamma^{\ l}}{\partial \alpha_j}\right)\right] \qquad (i = 1, \ldots, k;\ j = 1, \ldots, \varkappa)$$

$$B_{i,\varkappa+j} = M\left[\sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial a_j} + \sum_{\gamma=1}^{y} \frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\ l}} \frac{\partial \alpha_\gamma^{\ l}}{\partial a_j}\right) + \frac{\partial f_i}{\partial a_j}\right] (i=1, \ldots, k;\ j=1, \ldots, m)$$

$$B_{ij}^{\ l} = M\left[\left(\sum_{\nu=1}^{n} \frac{\partial f_{k+i}}{\partial X_{\nu 1}} \frac{\partial X_{\nu 1}}{\partial \alpha_j^{\ l}}\right) \Big|\, l\right] \qquad (i, j = 1, \ldots, y) \qquad (2.2)$$

$$\frac{\partial X_{\nu 1}}{\partial \alpha_j} = \int_{t_0^{\ l}}^{t_1^{\ l}} \sum_{s=1}^{r} \frac{\partial H^{(\nu)}}{\partial u_s} \delta u_{sj}\, dt, \qquad \frac{\partial X_{\nu 1}}{\partial \alpha_j^{\ l}} = \int_{t_0^{\ l}}^{t_1^{\ l}} \sum_{s=1}^{r_0} \frac{\partial H^{(\nu)}}{\partial u_s^{\ l}} \delta u_{sj}^{\ l}\, dt$$

$$\frac{\partial X_{\nu 1}}{\partial a_j} = \int_{t_0^{\ l}}^{t_1^{\ l}} \frac{\partial H^{(\nu)}}{\partial a_j}\, dt, \qquad H^{(\nu)} = \sum_{i=1}^{n} \Lambda_i^{(\nu)} \varphi_i(t, X, v, v^l, P, L)$$

In the above, Expressions $\Lambda_i^{(\nu)}, \partial\alpha_\gamma^{\ l}/\partial\alpha_j, \partial\alpha_\gamma^{\ l}/\partial a_j$ are defined by Equations

$$\Lambda_i^{(\nu)} = -\frac{\partial H^{(\nu)}}{\partial X_i}, \quad \Lambda_i^{(\nu)}(t_1^{\ l}) = \delta_{i\nu} \qquad (i, \nu = 1, \ldots, n)$$

$$\sum_{\nu=1}^{n} M\left[\left(\sum_{\gamma=1}^{y} \frac{\partial f_i}{\partial X_{\nu 1}} \frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\ l}} \frac{\partial \alpha_\gamma^{\ l}}{\partial \alpha_j} + \frac{\partial f_i}{\partial X_{\nu 1}} \frac{\partial X_{\nu 1}}{\partial \alpha_j}\right) \Big|\, l\right] = 0 \qquad (2.3)$$

$$\sum_{\nu=1}^{n} M\left[\left(\sum_{\gamma=1}^{y} \frac{\partial f_i}{\partial X_{\nu 1}} \frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\ l}} \frac{\partial \alpha_\gamma^{\ l}}{\partial a_s} + \frac{\partial f_i}{\partial X_{\nu 1}} \frac{\partial X_{\nu 1}}{\partial a_s} + \frac{\partial f_i}{\partial a_s}\right) \Big|\, l\right] = 0$$

$$(i = k+1, \ldots, k_0;\ j = 1, \ldots, \varkappa;\ s = 1, \ldots, m)$$

Here $\delta_{i\nu}$ is the Kronecker delta; $\delta u_{sj}(t)$, $\delta u_{sj}^{\ l}(t, L)$ are arbitrary functions from U.

T h e o r e m 2.2 . If the conditions of Theorem 2.1 are fulfilled, then there exists a vector function $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ and multipliers $\mu_0 = 1$, $\mu_i =$ const , $\mu_j = \mu_j(L)$ $(i = 1, \ldots, k;\ j = k+1, \ldots, k_0)$ relative to which the optimum control v, v^l, T^l, which maximizes (minimizes) functional (1.5) has the following properties:

a) It minimizes (maximizes) the function $M^\circ[H(t, X, v, v^l, P, L, \Lambda)]$ of the variable u for any of the t realized;

b) It minimizes (maximizes) the functions

$$M\,[H\ (t, X, v, v^l, P, L, \Lambda) \,|\, l]$$

of the variable u^l for any t, $t^l_{\ 0} \leqslant t \leqslant t^l_{\ 1}$ and $L \subset \Omega_l$;

c) It satisfies the relations

$$M\left[\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial a_j}-\int_{t_0^l}^{t_1^l}\frac{\partial H}{\partial a_j}\,dt\right]=0,\quad M\left[\left(\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial a_s^l}-\int_{t_0^l}^{t_1^l}\frac{\partial H}{\partial a_s^l}\,dt\right)\Big|\,l\right]=0 \tag{2.4}$$

$$M\left[\left(\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial t_0^l}+H\,|_{t_0^l}\right)\Big|\,l\right]=0,\qquad M\left[\left(\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{dt_1^l}-H\,|_{t_1^l}\right)\Big|\,l\right]=0$$

$$H=\Lambda_1\varphi_1+\ldots+\Lambda_n\varphi_n\qquad (j=1,\ldots,m;\ s=1,\ldots,m_0)$$

Here $M^\circ(H)$ denotes that part of the integral which determines $M(H)$, where t satisfies the inequality $t^l_0(l)\leqslant t\leqslant t^l_1(l)$; the random continuous vector-function $\Lambda=(\Lambda_1,\ldots,\Lambda_1)$ is determined by Equations

$$\Lambda_i^{\cdot}=-\frac{\partial H}{\partial X_i},\qquad \Lambda_i(t_1^l)=\Lambda_{i1}=-\sum_{j=0}^{k_0}\mu_j\frac{\partial f_j}{\partial X_{i1}}\qquad (i=1,\ldots,n) \tag{2.5}$$

3. Let us cite the schemes which can be used to prove Theorems 2.1 and 2.2. Let v, v^l, T^l be the permissible control of system (1.2) to (1.5). We shall now consider the control

$$(v^*,\ v^{l*},\ T^{l*})=[u^*=u(t)+\delta u(t),\quad a^*=a+\delta a,\quad u^{l*}=u^l(t,L)+\delta u^l(t,L)$$
$$a^{l*}=a^l(L)+\delta a^l(L),\qquad T^{l*}=T^l(L)+\delta T^l(L)]$$

from U. Here $u=[u_i(t)],\ u^l=[u^l_j\ (t,L)]\ (i=1,\ldots,r,\ j=1,\ldots,r_0)$ for any fixed L are considered supplemented in their definition beyond the limits of the segments $[t^l_0,t^l_1]$ with preservation of the continuity and continuous differentiability at the points $t=t^l_0,\ t=t^l_1$; $\delta u_j,\ \delta u^l_j$ are of the form

$$\delta u_j=\sum_{i=1}^{\varkappa}\alpha_i\delta u_{ji}(t),\qquad \delta u_j^l=\sum_{i=1}^{k_0-k}\alpha_i^l(L)\,\delta u_{ji}^l(t,L)$$

everywhere except an arbitrarily small segment $[t',\ t'']$, $t''-t'=\tau\geq 0$, where

$$\delta u_j=\omega_j-u_j,\quad \delta u_j^l=\omega_j^l-u_j^l;\quad \delta u_{ji}(t),\quad \delta u_{ji}^l(t,L)$$
$$\omega=[\omega_i(t)],\quad \omega^l=[\omega_i^l(t,L)],\quad \delta a^l(L),\quad \delta T^l(L)$$

are arbitrary functions of the corresponding arguments, and u,ω,u^l,ω^l, are considered continuous on the interval $(t',\ t'')$;

$$\delta a,\ \delta a^l,\ \delta T^l,\ \tau,\ \alpha=(\alpha_1,\ldots,\alpha_\varkappa),\ \alpha^l=(\alpha_1^l,\ldots,\alpha^l_{k_0-k})$$

are arbitrary vectors.

Substituting $v^*,\ v^{l*},\ T^{l*}$ in (1.2) we find the trajectory $X^*=(X^*_1,\ldots\ldots,X_n^*)$, where X_i^* are known functions of the arguments

$$t,\ \gamma=(\gamma_i)\equiv(\alpha_i,\ \alpha^l{}_i,\ \delta a_i,\ \delta a_i^l,\ \delta t^l{}_i,\ \tau).$$

The control $v^*,\ v^{l*},\ T^{l*}$ can be considered permissible by definition if γ_i satisfy Equations

$$M\,[f_i(X_1^*,\ a^*,\ a^{l*},\ T^{l*},\ P,\ L)]=0\qquad (i=1,\ldots,k) \tag{3.1}$$
$$M\,[f_i(X_1^*,\ a^*,\ a^{l*},\ T^{l*},\ P,\ L)\,|\,l]=0\quad (i=k+1,\ldots,k_0)\qquad (X_1^*=X^*(t_1^{l*})) \tag{3.2}$$

Equations (3.1) and (3.2) have the solution $\gamma=0$, since $v,\ v^l,\ T^l$ is a permissible control. By virtue of the agreed assumptions about the properties of the functions $f_i,\varphi_i,\psi_i,u_i^*,u^l{}_i^*$, the rules of supplementary definition of u,u^l beyond the limits of the segments $[t^l_0,t^l_1)$ and by the isolation of the interval $(t',\ t'')$, their left sides are continuous and continuously differentiable with respect to γ_i. Hence, the theorem on the existence of the implicit functions of Equation (3.2) can be used to determine the continuous and continuously differentiable functions $\alpha^l{}_j\ (j=1,\ldots,\nu)$ of the variables α_j, δa_j, $\delta a^l{}_j$, $\delta t^l{}_j$, τ, L, provided the rank of the matrix

$$A^{l*} = \left[M \left(\frac{\partial f_{k+i}}{\partial \alpha_j{}^l} \,\Big|\, l \right) \right]_{\gamma=0} \qquad (i, j = 1, \ldots, y)$$

is $y = k_0 = k$.

In precisely the same way system (3.1), where $\alpha^l{}_j$ $(j = 1, \ldots, y)$ are defined by Equations (3.2), determines the implicit functions α_j, δa_j if the rank of the matrix

$$A^* = \left\| \begin{matrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & M\left[\dfrac{\partial f_i}{\partial \alpha_j}\right]_{\gamma=0} & \cdots & M\left[\dfrac{\partial f_i}{\partial a_s}\right]_{\gamma=0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix} \right\| \qquad \begin{pmatrix} i = 1, \ldots, k \\ j = 1, \ldots, \varkappa \\ s = 1, \ldots, m \end{pmatrix}$$

is k .

Writing out the expressions for the elements of the matrices A^{l*}, A^*, we see the validity of the identities $A^* \equiv A$, $A^{l*} = A^l$, where A and A^l are defined by relations (2.1) to (2.3). Thus, there exist γ_i not simultaneously equal to zero which satisfy Equations (3.1) and (3.2) provided the ranks of the matrices A and A^l are k and y , respectively. But in this case there exists a permissible control $(v^*, v^{l*}, T^{l*}) \neq (v, v^l, T^l)$, which contains v, v^l, T^l for $\gamma = 0$. Theorem 2.1 has been proved.

Let us suppose now that v, v^l, T^l is a permissible control which satisfies the conditions of Theorem 2.1 and maximizes functional (1.5). There then exists a family of permissible controls in which

$$\Delta J_i = M\,[f_i(\mathbf{X}_1^*, a^*, a^{l*}, T^{l*}, P, L) - f_i(\mathbf{X}_1, a, a^l, T^l, P, L)] = \tag{3.3}$$

$$= \sum_{j=1}^{\varkappa} B_{ij}\alpha_j + \sum_{j=1}^{m} B_{i,\,\varkappa+j}\delta a_j + \sum_{j=0}^{1} B_{ij}{}^t + \sum_{j=1}^{m_0} B_{ij}{}^a + B_i\tau + \varepsilon$$

$$(i = 0, \ldots, k)$$

$$B_{ij} = M\left[\sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial \alpha_j} + \sum_{\gamma=1}^{k_0-k} \frac{\partial X_{1\nu}}{\partial \alpha_\gamma{}^l}\frac{\partial \alpha_\gamma{}^l}{\partial \alpha_j}\right)\right]$$

$$B_{i,\,\varkappa+j} = M\left[\sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial a_j} + \sum_{\gamma=1}^{k_0-k} \frac{\partial X_{\nu 1}}{\partial \alpha_\gamma{}^l}\frac{\partial \alpha_\gamma{}^l}{\partial a_j}\right) + \frac{\partial f_i}{\partial a_j}\right]$$

$$B_{ij}{}^a = M\left\{\left[\sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial a_j{}^l} + \sum_{\gamma=1}^{k_0-k} \frac{\partial X_{\nu 1}}{\partial \alpha_\gamma{}^l}\frac{\partial \alpha_\gamma{}^l}{\partial a_j{}^l}\right) + \frac{\partial f_i}{\partial a_j{}^l}\right]\delta a_j{}^l\right\} \tag{3.4}$$

$$B_{ij}{}^t = M\left\{\left[\sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial t_j{}^l} + \sum_{\gamma=1}^{k_0-k} \frac{\partial X_{\nu 1}}{\partial \alpha_\gamma{}^l}\frac{\partial \alpha_\gamma{}^l}{\partial t_j{}^l}\right) + \frac{\partial f_i}{\partial t_j{}^l}\right]\delta t_j{}^l\right\}$$

$$B_i = M\left[\sum_{\nu=1}^{n} \frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial \tau} + \sum_{\gamma=1}^{k_0-k} \frac{\partial X_{\nu 1}}{\partial \alpha_\gamma{}^l}\frac{\partial \alpha_\gamma{}^l}{\partial \tau}\right)\right]$$

$$\Delta J_0 = \Delta J \leqslant 0, \qquad \Delta J_i = 0 \quad (i = 1, \ldots, k)$$

Here ε is a quantity of higher than the first order of smallness; the coefficients of α_i, δa_i, $\delta a_i{}^l$, $\delta t_i{}^l$, τ are computed for $\gamma = 0$. The functions $\alpha_\gamma{}^l$ of the variables α_i, δa_i, $\delta a_i{}^l$, $\delta t_i{}^l$, τ are determined by Equations (3.2), so that the partial derivatives

$$\frac{\partial \alpha_\gamma{}^l}{\partial \alpha_j}, \quad \frac{\partial \alpha_\gamma{}^l}{\partial a_j} = \frac{\partial \alpha_\gamma{}^l}{\partial \delta a_j}, \quad \frac{\partial \alpha_\gamma{}^l}{\partial t_j{}^l} = \frac{\partial \alpha_\gamma{}^l}{\partial \delta t_j{}^l}, \quad \frac{\partial \alpha_\gamma{}^l}{\partial \tau}$$

satisfy Equations

$$M\left\{\left[\sum_{\nu=1}^{n}\frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial \alpha_j}+\sum_{\gamma=1}^{k_0-k}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^l}\frac{\partial \alpha_\gamma^l}{\partial \alpha_j}\right)\right]\Big| l\right\}=0 \qquad \begin{pmatrix} i=k+1,\ldots,k_0\\ j=1,\ldots,\varkappa\end{pmatrix} \tag{3.5}$$

$$M\left\{\left[\sum_{\nu=1}^{n}\frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial a_j}+\sum_{\gamma=1}^{k_0-k}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^l}\frac{\partial \alpha_\gamma^l}{\partial a_j}\right)+\frac{\partial f_i}{\partial a_j}\right]\Big| l\right\}=0 \qquad \begin{pmatrix} i=k+1,\ldots,k_0\\ j=1,\ldots,m\end{pmatrix}$$

$$M\left\{\left[\sum_{\nu=1}^{n}\frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial a_j^l}+\sum_{\gamma=1}^{k_0-k}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^l}\frac{\partial \alpha_\gamma^l}{\partial a_j^l}\right)+\frac{\partial f_i}{\partial a_j^l}\right]\Big| l\right\}=0 \qquad \begin{pmatrix} i=k+1,\ldots,k_0\\ j=1,\ldots,m_0\end{pmatrix}$$

$$M\left\{\left[\sum_{\nu=1}^{n}\frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial t_j^l}+\sum_{\gamma=1}^{k_0-k}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^l}\frac{\partial \alpha_\gamma^l}{\partial t_j^l}\right)+\frac{\partial f_i}{\partial t_j^l}\right]\Big| l\right\}=0 \qquad \begin{pmatrix} i=k+1,\ldots,k_0\\ j=0,1\end{pmatrix}$$

$$M\left\{\left[\sum_{\nu=1}^{n}\frac{\partial f_i}{\partial X_{\nu 1}}\left(\frac{\partial X_{\nu 1}}{\partial \tau}+\sum_{\gamma=1}^{k_0-k}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^l}\frac{\partial \alpha_\gamma^l}{\partial \tau}\right)\right]\Big| l\right\}=0 \qquad (i=k+1,\ldots,k_0)$$

It is evident that the following relations are valid:

$$\frac{\partial X_{\nu 1}}{\partial \alpha_j}=\int_{t_0^l}^{t_1^l}\sum_{s=1}^{r}\frac{\partial H^{(\nu)}}{\partial u_s}\delta u_{sj}dt, \qquad \frac{\partial X_{\nu 1}}{\partial a_j}=\int_{t_0^l}^{t_1^l}\frac{\partial H^{(\nu)}}{\partial a_j}dt$$

$$\frac{\partial X_{\nu 1}}{\partial \alpha_j^l}=\int_{t_0^l}^{t_1^l}\sum_{s=1}^{r_0}\frac{\partial H^{(\nu)}}{\partial u_s^l}\delta u_{sj}^l dt, \qquad \frac{\partial X_{\nu 1}}{\partial a_j^l}=\int_{t_0^l}^{t_1^l}\frac{\partial H^{(\nu)}}{\partial a_j^l}dt \tag{3.6}$$

$$\frac{\partial X_{\nu 1}}{\partial \tau}=[H^{(\nu)*}-H^{(\nu)}]_{t'} \quad (t_0^l\leqslant t'<t_1^l), \qquad \frac{\partial X_{\nu 1}}{\partial \tau}=0 \quad (t_0^l>t',\ t_1^l\leqslant t')$$

$$\frac{\partial X_{\nu 1}}{\partial t_0^l}=-H^{(\nu)}|_{t_0^l}, \qquad \frac{\partial X_{\nu 1}}{\partial t_1^l}=H^{(\nu)}|_{t_1^l}$$

Here

$$H^{(\nu)}=\sum_{i=1}^{n}\Lambda_i^{(\nu)}\varphi_i(t,X,v,v^l,P,L), \qquad H^{(\nu)*}=\sum_{i=1}^{n}\Lambda_i^{(\nu)}\varphi_i(t,X,\omega,a,\omega^l,a^l,P,L)$$

$$\dot\Lambda_i^{(\nu)}=-\frac{\partial H^{(\nu)}}{\partial X_i}, \qquad \Lambda_j^{(\nu)}(t_1^l)=\delta_{i\nu} \qquad (i,\nu=1,\ldots,n) \tag{3.7}$$

Taking account of (3.3), we obtain the required condition for the maximum of functional (1.5),

$$\delta J=\sum_{j=1}^{\varkappa}\sum_{i=0}^{k}\mu_i B_{ij}\alpha_j+\sum_{j=1}^{m}\sum_{i=0}^{k}\mu_i B_{i,\varkappa+j}\delta a_j+\sum_{j=0}^{1}\sum_{i=0}^{k}\mu_i B_{ij}^{\,l}+$$
$$+\sum_{j=1}^{m_0}\sum_{i=0}^{k}\mu_i B_{ij}^{\,a}+\sum_{i=0}^{k}\mu_i B_i\tau\leqslant 0 \tag{3.8}$$

Here $\mu_0=1$, $\mu_1,\ldots,\mu_k$ are constant nonrandom Lagrange multipliers. Since the condition of Theorem 2.1 as regards the rank of the matrix A is assumed to be fulfilled, the multipliers $\mu_1,\ldots,\mu_k$ can be chosen on the basis of the condition whereby k coefficients of α_j, δa_j in the right-hand side of (3.8) vanish. The remaining term in (3.8) is independent and arbitrary. Hence, taking account of the inequality $\tau\geq 0$, we have instead of (3.8) that

$$\sum_{i=0}^{k}\mu_i B_{ij}=0, \quad \sum_{i=0}^{k}\mu_i B_{i\nu}^{\,a}=0, \quad \sum_{i=0}^{k}\mu_i B_{is}^{\,l}=0, \quad \sum_{i=0}^{k}\mu_i B_i\leqslant 0 \tag{3.9}$$
$$(j=1,\ldots,\varkappa+m;\ s=0,1;\ \nu=1,\ldots,m_0)$$

Let us multiply each equation in (3.5) with the subscript i by some indefinite Lagrange multiplier $\mu_i(L)$ $(i = k+1,\ldots,k_0)$, find the mathematical expectation of the results of multiplication, and sum them over i for a fixed j in each group of equations. Adding the resulting expressions which are equal to zero to the left-hand sides of relations (3.9), we find that

$$M\Big[\sum_{i=0}^{k_0}\sum_{\nu=1}^{n}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial \alpha_j}+\sum_{\gamma=1}^{k_0-k}\sum_{\nu=1}^{n}\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\,l}}\frac{\partial \alpha_\gamma^{\,l}}{\partial \alpha_j}\Big]=0$$
$$(j=1,\ldots,\varkappa)$$

$$M\Big[\sum_{i=0}^{k_0}\sum_{\nu=1}^{n}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial a_j}+\sum_{\gamma=1}^{k_0-k}\sum_{\nu=1}^{n}\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\,l}}\frac{\partial \alpha_\gamma^{\,l}}{\partial a_j}+\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial a_j}\Big]=0$$
$$(j=1,\ldots,m)$$

$$M\Big[\sum_{i=0}^{k_0}\sum_{\nu=1}^{n}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial a_j^{\,l}}+\sum_{\gamma=1}^{k_0-k}\sum_{\nu=1}^{n}\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\,l}}\frac{\partial \alpha_\gamma^{\,l}}{\partial a_j^{\,l}}+\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial a_j^{\,l}}\Big]=0$$
$$(j=1,\ldots,m_0) \qquad (3.10)$$

$$M\Big[\sum_{i=0}^{k_0}\sum_{\nu=1}^{n}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial t_j^{\,l}}+\sum_{\gamma=1}^{k_0-k}\sum_{\nu=1}^{n}\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\,l}}\frac{\partial \alpha_\gamma^{\,l}}{\partial t_j^{\,l}}+\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial t_j^{\,l}}\Big]=0$$
$$(j=0,1)$$

$$M\Big[\sum_{i=0}^{k_0}\sum_{\nu=1}^{n}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial \tau}+\sum_{\gamma=1}^{k_0-k}\sum_{\nu=1}^{n}\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\,l}}\frac{\partial \alpha_\gamma^{\,l}}{\partial \tau}\Big]\leqslant 0$$

The condition of Theorem 2.1 as regards the rank of the matrix ~ is fulfilled, so that the quantities $\mu_i(L)$ $(i = k+1,\ldots,k_0)$ can be found from Equations

$$M\Big[\Big(\sum_{i=0}^{k_0}\sum_{\nu=1}^{n}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}\frac{\partial X_{\nu 1}}{\partial \alpha_\gamma^{\,l}}\Big)\Big|\,l\Big]=0 \qquad (\gamma=1,\ldots,k_0-k) \qquad (3.11)$$

We set

$$\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}=-\Lambda_{\nu 1}, \qquad \sum_{\nu=1}^{n}\Lambda_i^{(\nu)}\Lambda_{\nu 1}=\Lambda_i \qquad (\gamma, i=1,\ldots,n)$$

Then, taking account of (3.7), we have

$$\sum_{\nu=1}^{n}\Lambda_{\nu 1}H^{(\nu)}=\sum_{i=1}^{n}\Lambda_i\varphi_i=H, \quad \Lambda_i^{\cdot}=-\frac{\partial H}{\partial X_i}, \qquad \Lambda_\nu(t_1^{\,l})=\Lambda_{\nu 1}=-\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial X_{\nu 1}}$$
$$(i,\nu=1,\ldots,n)$$

Relations (3.10) and (3.11) can now be written as

$$M\Big[\int_{t_0^{\,l}}^{t_1^{\,l}}\sum_{s=1}^{r}\frac{\partial H}{\partial u_s}\,\delta u_{sj}\,dt\Big]=0, \qquad M\Big[\Big(\int_{t_0}^{t_1^{\,l}}\sum_{s=1}^{r_0}\frac{\partial H}{\partial u_s^{\,l}}\,\delta u_{si}^{\,l}\,dt\Big)\Big|\,l\Big]=0$$
$$(j=1,\ldots,\varkappa;\quad i=1,\ldots,k_0-k) \qquad (3.12)$$

$$M\Big\{\Big[\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial t_0^{\,l}}+H\,|_{t_0^{\,l}}\Big]\delta t_0^{\,l}\Big\}=0, \qquad M\Big\{\Big[\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial t_1^{\,l}}-H\,|_{t_1^{\,l}}\Big]\delta t_1^{\,l}\Big\}=0$$

$$M\Big\{\Big[\sum_{i=0}^{k_0}\mu_i\frac{\partial f_i}{\partial a_j^{\,l}}-\int_{t_0^{\,l}}^{t_1^{\,l}}\frac{\partial H}{\partial a_j^{\,l}}\,dt\Big]\delta a_j^{\,l}\Big\}=0 \qquad (j=1,\ldots,m_0) \qquad (3.13)$$

$$M\left[\sum_{i=0}^{k_0} \mu_i \frac{\partial f_i}{\partial a_j} - \int_{t_0^l}^{t_1^l} \frac{\partial H}{\partial a_j} dt\right] = 0 \quad (j = 1, \ldots, m) \tag{3.14}$$

$$M^\circ [H^* - H]_{l'} \geqslant 0, \qquad H^* = \sum_{i=1}^{n} \Lambda_i \varphi_i (t, X, \omega, a, \omega^l, a^l, P, L) \tag{3.15}$$

Inequality (3.15) follows from the last relation of system (3.10). In fact, the triple sum in square brackets in this relation is equal to zero by virtue of (3.11), while the remaining terms vanish for all l, where $t_0^l(l) \geqslant t'$, $t_1^l(l) \leqslant t'$, since here $\partial X_{\nu 1} / \partial \tau = 0$ (see (3.5)).

It is clear that relations (3.13) can be written as

$$\int_{\Omega_l} M\left[\left(\sum_{i=0}^{k_0} \mu_i \frac{\partial f_i}{\partial t_0^l} + H|_{t_0^l}\right) \Big| \, l\right] f_l(l)\, \delta t_0^l(l)\, dl = 0$$

$$\int_{\Omega_l} M\left[\left(\sum_{i=0}^{k_0} \mu_i \frac{\partial f_i}{\partial t_1^l} - H|_{t_1^l}\right) \Big| \, l\right] f_l(l)\, \delta t_1^l(l)\, dl = 0$$

$$\int_{\Omega_l} M\left[\left(\sum_{i=0}^{k_0} \mu_i \frac{\partial f_i}{\partial a_j^l} - \int_{t_0^l}^{t_1^l} \frac{\partial H}{\partial a_j^l} dt\right) \Big| \, l\right] f_l(l)\, \delta a_j^l(l)\, dl = 0 \qquad (j = 1, \ldots, m_0)$$

Hence, taking account of the arbitrariness of the functions $\delta t_0^l(l)$, $\delta t_1^l(l)$, $\delta a_j^l(l)$, we obtain

$$M\left[\left(\sum_{i=0}^{k_0} M_i \frac{\partial f_i}{\partial t_0^l} + H|_{t_0^l}\right) \Big| \, l\right] = 0, \qquad M\left[\left(\sum_{i=0}^{k_0} \mu_i \frac{\partial f_i}{\partial t_1^l} - H|_{t_1^l}\right) \Big| \, l\right] = 0 \tag{3.16}$$

$$M\left[\left(\sum_{i=0}^{k_0} \mu_i \frac{\partial f_i}{\partial a_j^l} - \int_{t_0^l}^{t_1^l} \frac{\partial H}{\partial a_j^l} dt\right) \Big| \, l\right] = 0 \quad (j = 1, \ldots, m_0)$$

Relations (3.14) and (3.16) coincide with conditions (c) of Theorem 2.2. The validity of conditions (a) and (b) of this theorem follows from inequality (3.15). In fact, since ω and ω^l are independent, we find from (3.15) that

$$M^\circ [H(t, X, \omega, a, u^l, a^l, P, L, \Lambda) - H(t, X, u, a, u^l, a^l, P, L, \Lambda)]_{t'} \geqslant 0 \tag{3.17}$$

$$M^\circ [H(t, X, u, a, \omega^l, a^l, P, L, \Lambda) - H(t, X, u, a, u^l, a^l, P, L, \Lambda)]_{t'} \geqslant 0 \tag{3.18}$$

(3.17) implies condition (a) of Theorem 2.2 directly.

Let us suppose there is a situation contrary to that stipulated in condition (b) of Theorem 2.2, i.e. that there is a point $(L, t) = (l', t')$ where

$$M\{[H(t, X, u, a, \omega^l, a^l, P, L, \Lambda) - H(t, X, u, a, u^l, a^l, P, L, \Lambda)] \,|\, l\}_{l', t'} < 0 \tag{3.19}$$

Then, by virtue of the continuity of H with respect to ω^l and the piecewise continuity of $\omega^l(t, L)$, it is possible to find a segment $[l^*, l^{**}]$ which includes the point $L = l'$, where the inequality sign in (3.19) remains unchanged. The function ω^l can be chosen in such a way that it differs from u^l only on the segment $[l^*, l^{**}]$.

Hence we have

$$M^\circ [H^* - H]_{t'} = \int_{l^*}^{l^{**}} M\{[H(t, X, u, a, \omega^l, a^l, P, L, \Lambda) -$$

$$- H(t, X, u, a, u^l, a^l, P, L, \Lambda)] \,|\, l\}_{t'} f_l(l)\, dl < 0$$

which contradicts condition (3.18). It remains for us to accept the validity of condition (b) of Theorem 2.2 and thereby the validity of all the statements comprising the theorem. In conclusion we note that Equations (3.12) are satisfied identically by virtue of condition (a) of Theorem 2.2.

4. In addition to solving several new problems in statistical dynamics, the above results also make it possible to formulate a new procedure for synthesizing program control systems which, in contrast to the existing techniques, enables to solve a problem "as a whole" when the programed motion and the transient process control law are optimized from general methodological standpoint with allowance for their interrelationships with respect to a single criterion. The scope of the present paper requires that we explain only its basic features as illustrated by the following highly specific example.

Let there be a controlled object

$$X^{\cdot} = aX + BW, \qquad X(t_0) = X_0, \qquad X(t_1) = C_1, \quad t_0 \leqslant t \leqslant t_1 \tag{4.1}$$

Here a, C_1, t_0, t_1 are known numbers; B, X_0 are specified continuous random quantities with the mathematical expectations m_b, m_{x_0}; W is the control signal.

We know that if the control objective is achieved under the programed conditions, the W can be represented in the form $W = u + u^l$, where $u = u(t)$ is the constantly realized programed part of W, and $u^l = u^l(t, X_2, \ldots)$ is the control signal whose purpose is to compensate random disturbances, which in our case happen to be the deviations $B^0 = B - m_b$, $X_0{}^0 = X_0 - m_{x0}$. The trajectory of the object X is here subdivided into the programed component X_1 and the disturbed component $X_2 = X - X_1$, which are given by Equations

$$X_1^{\cdot} = aX_1 + m_b\, u, \qquad X_1(t_0) = m_{x0}, \qquad X_1(t_1) = M(X_{11}) = C_1 \tag{4.2}$$

$$X_2^{\cdot} = aX_2 + Bu^l + B^\circ u, \qquad X_2(t_0) = X_0{}^0, \qquad X_2(t_1) = M(X_{21} \mid b, x_0) = 0 \tag{4.3}$$

Existing methods of synthesizing u, u^l are characterized by the choice of programed notion independently of the disturbed motion. This opens the way for inadmissible solutions. For example, let a control process be optimized with respect to the energy expenditure described by the functionals

$$J_0 = g_0 \int_{t_0}^{t_1} u^2 dt, \qquad J_1 = g_1 \int_{t_0}^{t_1} u^{l2}\, dt, \qquad g_i = \text{const} > 0 \quad (i=0,1)$$

in the programed and disturbed motions, respectively. Independent minimization of these functions generally does not maximize the total energy expenditure, since, as is evident from (4.3), the characteristics of disturbed motion depend on the programed motion. It is more expedient, therefore, to optimize u, u^l by minimizing a functional of the form

$$J = M\left[g_0 \int_{t_0}^{t_1} u^2\, dt + g_1 \int_{t_0}^{t_1} u^{l2} dt\right] \tag{4.4}$$

or by some other criterion which affords a notion of the overall energy loss.

We shall show that the optimization of u, u^l relative to criteria of the form (4.4) can be attained by the methods of the theory we are developing. We introduce the function $X_3(t)$ defined by Equations

$$X_3^{\cdot} = g_0 u^2 + g_1 u^{l2}, \qquad X_3(t_0) = 0 \tag{4.5}$$

Then instead of (4.4) we can write

$$J = M(X_{31}) \tag{4.6}$$

Now the problem of synthesizing control of object (4.1) reduces to the solution of the following optimum problem: for system (4.2), (4.3), (4.5) we are to find the control $u(t)$, $u^l(t, X_2, \ldots)$, which minimizes functional (4.6). It is clear that this problem is of the form considered in Section 1, where

$$n = 3, \quad k = 1, \quad k_0 = 2, \quad L = (L_i) \equiv (B,\ X_0), \quad t_0{}^l = t_0, \quad t_1{}^l = t_1$$
$$\varphi_1 = aX_1 + m_b u, \quad \varphi_2 = aX_2 + Bu^l + B^\circ u, \quad \varphi_3 = g_0 u^2 + g_1 u^{l2}$$
$$f_0 = X_{31}, \quad f_1 = X_{11} - c_1, \quad f_2 = X_{21}$$

It can be solved by applying the conditions of Theorem 2.2, whereby we have

$$H = \Lambda_1 (aX_1 + m_b u) + \Lambda_2 (aX_2 + Bu^l + B^\circ u) + \Lambda_3 (g_0 u^2 + g_1 u^{l2})$$

$$\Lambda_1^{\cdot} = -\frac{\partial H}{\partial X_1} = -a\Lambda_1, \qquad \Lambda_1(t_1) = -\sum_{i=0}^{2} \mu_i \frac{\partial f_i}{\partial X_{11}} = -\mu_1 = \mathrm{const}$$

$$\Lambda_2^{\cdot} = -\frac{\partial H}{\partial X_2} = -a\Lambda_2, \qquad \Lambda_2(t_1) = -\sum_{i=0}^{2} \mu_i \frac{\partial f_i}{\partial X_{21}} = -\mu_2 (B,\ X_0)$$

$$\Lambda_3^{\cdot} = -\frac{\partial H}{\partial X_3} = 0, \qquad \Lambda_3(t_1) = -\sum_{i=0}^{2} \mu_i \frac{\partial f_i}{\partial X_{31}} = -\mu_0 = -1$$

$$M[m_b\Lambda_1 + B^\circ\Lambda_2 + 2g_0\Lambda_3 u] = 0, \qquad B\Lambda_2 + 2g_1\Lambda_3 u^l = 0$$

Hence we obtain

$$u = \frac{M(m_b\Lambda_1 + B^0\Lambda_2)}{2g_0} = -\frac{m_b\mu_1 + M(B^0\mu_2)}{2g_0} e^{a(t_1-t)} \tag{4.7}$$

$$u^l = \frac{B\Lambda_2}{2g_1} = -\frac{B\mu_2}{2g_1} e^{a(t_1-t)} \tag{4.8}$$

Substituting Expressions (4.7) and (4.8) in the right-hand side of (4.3), integrating, and satisfying the boundary condition $X_{21} = 0$, we find that

$$X_0{}^0 e^{a(t_1-t_0)} + \frac{1 - e^{2a(t_1-t_0)}}{4a}\left[\frac{B^0 m_b\mu_1 + B^0 M(B^0\mu_2)}{g_0} + \frac{B^2\mu_2}{g_1}\right] = 0 \tag{4.9}$$

From this we have

$$M(B^0\mu_2) = \frac{g_0 D_0 - g_1 m_b D_1 \mu_1}{g_0 + g_1 D_1} \tag{4.10}$$

where

$$D_0 = -\frac{4ag_1 e^{a(t_1-t_0)}}{1 - e^{2a(t_1-t_0)}} M\left(\frac{B^0 X_0{}^0}{B^2}\right), \qquad D_1 = M\left(\frac{B^{02}}{B^2}\right)$$

Integration of Equation (4.2) with allowance for Expressions (4.7) and (4.8) and subsequent satisfaction of the boundary condition $X_{1*} = C_1$ makes it possible to write the following expression for the multiplier μ_1:

$$\mu_1 = \frac{g_0 + g_1 D_1}{m_b}\left[\frac{4a(c_1 - m_{x0} e^{a(t_1-t_0)})}{m_b(1 - e^{2a(t_1-t_0)})} - \frac{D_0}{g_0 + g_1 D_1}\right] \tag{4.11}$$

Computing μ_1, $M(B^0\mu_2)$ from Formulas (4.10) and (4.11) and substituting the results into Formula (4.7) we find the optimum program of motion $u(t)$. The optimum control law $u^l = u^l(t, X_2, \ldots)$ of the disturbed motion X_2 can be found from (4.9) with allowance for (4.8) upon replacement of $X_2^0 = X_0^0$, t_0 in (4.9) by the instantaneous values of X_2, t in the form

$$u^e = \frac{e^{a(t_1-t)}}{2B}\left[\frac{4aX_2 e^{a(t_1-t)}}{1 - e^{2a(t_1-t)}} + \frac{B^0(m_b\mu_1 + M(B^0\mu_2))}{g_0}\right] \tag{4.12}$$

In conclusion we note that the principle of average optimality implies the conditions of determined systems optimization as the limiting cases when the domains of realization of the random parameters are contracted to points. Among these implications is the maximum principle as formulated by L.S.Pontriagin.

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